

Naval Research Laboratory

Washington, DC 20375-5000



NRL Memorandum Report 6583

AD-A217 063

Elastic Solutions in a Semi-Infinite Solid With an Ellipsoidal Inclusion

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January 25, 1990

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90 01 22 1 35

REPORT DOCUMENTATION PAGE				Form Approved OMB No 0704-0188	
1a REPORT SECURITY CLASSIFICATION UNCLASSIFIED			1b RESTRICTIVE MARKINGS		
2a SECURITY CLASSIFICATION AUTHORITY			3 DISTRIBUTION/AVAILABILITY OF REPORT Approved for public release; distribution unlimited.		
2b DECLASSIFICATION/DOWNGRADING SCHEDULE					
4 PERFORMING ORGANIZATION REPORT NUMBER(S) NRL Memorandum Report 6583			5 MONITORING ORGANIZATION REPORT NUMBER(S)		
6a. NAME OF PERFORMING ORGANIZATION Naval Research Laboratory		6b OFFICE SYMBOL (If applicable) Code 6372	7a. NAME OF MONITORING ORGANIZATION		
6c ADDRESS (City, State, and ZIP Code) Washington, DC 20375-5000			7b ADDRESS (City, State, and ZIP Code)		
8a. NAME OF FUNDING/SPONSORING ORGANIZATION Office of Naval Research		8b OFFICE SYMBOL (If applicable)	9 PROCUREMENT INSTRUMENT IDENTIFICATION NUMBER		
8c ADDRESS (City, State, and ZIP Code) Arlington, VA 22217			10 SOURCE OF FUNDING NUMBERS		
			PROGRAM ELEMENT NO 61153N	PROJECT NO RR02204	TASK NO 41
					ORDERING AND ACQUISITION NO DN280-063
11 TITLE (Include Security Classification) Elastic Solutions in a Semi-Infinite Solid with an Ellipsoidal Inclusion					
12 PERSONAL AUTHOR(S) Yu,* H.Y. and Sanday, S.C.					
13a. TYPE OF REPORT Final		13b TIME COVERED FROM 1/89 TO 10/89		14 DATE OF REPORT (Year, Month, Day) 1990 January 25	
15 PAGE COUNT 19					
16 SUPPLEMENTARY NOTATION *Geo-Center, Inc., Fort Washington, MD 20744					
17 COSATI CODES			18 SUBJECT TERMS (Continue on reverse if necessary and identify by block number)		
FIELD	GROUP	SUB-GROUP	Elastic solution Orthotropic & shear misfit		
			Half space Inclusion		
			Image stress		
19 ABSTRACT (Continue on reverse if necessary and identify by block number) The elastic field caused by an ellipsoidal inclusion in a semi-infinite solid is investigated. The solutions for the inclusion with misfit stress-free strain components e_{ij}^T with $e_{11}^T = e_{22}^T$ and $e_{12}^T = 0$ are solved by the application of the Hankel transformation method for the solution of axisymmetric problems together with Mindlin's solution for the nuclei of strain in the semi-infinite solid and Eshelby's solution for ellipsoidal inclusions. The resultant stresses have the property that the traction vanishes across the plane boundary of the solid.					
20 DISTRIBUTION/AVAILABILITY OF ABSTRACT <input checked="" type="checkbox"/> UNCLASSIFIED/UNLIMITED <input type="checkbox"/> SAME AS RPT <input type="checkbox"/> DTIC USERS			21 ABSTRACT SECURITY CLASSIFICATION UNCLASSIFIED		
22a NAME OF RESPONSIBLE INDIVIDUAL S.C. Sanday			22b TELEPHONE (Include Area Code) (202) 767-2264		22c DDC CODE Code 6370

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DTIC TAB	<input type="checkbox"/>
Unannounced	<input type="checkbox"/>
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Dist	Avail and/or Special
A-1	

ELASTIC SOLUTIONS IN A SEMI-INFINITE SOLID WITH AN ELLIPSOIDAL INCLUSION

1. INTRODUCTION

The elastic fields due to inclusions in infinite media have been extensively investigated using Eshelby's method (1957, 1959, 1961). Cases of more practical interest are the elastic fields due to inclusion in semi-infinite media since many sites of initial strains, resulting from high stresses due to contact, heating, or metallurgical transformations, are confined to a near surface zone. The problem of an ellipsoidal inclusion which has undergone a simple shear is of interest in connection with twinning and martensitic or other diffusionless transformations. The elastic solution for an inclusion near the free surface has been solved for a spherical inclusion with pure dilatational eigenstrain (stress free transformation strain) (Mindlin and Cheng, 1950B), an ellipsoidal inclusion with pure dilatational eigenstrains (Seo and Mura, 1979) and a cuboidal inclusion with uniform eigenstrains (Chiu, 1977). In both Mindlin and Cheng's, and Seo and Mura's analyses, the solutions are obtained by integrating the Green's function of a point force in the interior of a semi-infinite solid (Mindlin, 1936, 1953). In Chiu's analysis, the free surface condition is satisfied by superimposing the solution of a half-space under normal surface traction on the full space solution due to a cuboidal inclusion and its image with the uniform eigenstrains.

The solution of the linear equations of equilibrium of an elastic body with a force acting at a point within an isotropic body bounded by a plane has been solved by starting with Kelvin's solution for a force in an infinite body and guessing the nuclei of strain to add outside of the semi-infinite body so as to annul the tractions on the plane boundary (Mindlin, 1936). The same results have been obtained directly by means of an application of potential theory (Mindlin, 1953). The stress functions of different strain of nuclei in the semi-infinite elastic solid (Mindlin and Cheng, 1950A) are derived, by the processes of superposition, differentiation and integration, from the solution for the single force in the interior of the semi-infinite solid.

The method of Hankel transformations, elaborated for cylindrically symmetrical problems of the theory of elasticity in Sneddon's book (1951), has been used to solve the stress field of a circular edge dislocation loop with Burger's vector normal to the plane of the loop (prismatic loop) in an unbounded solid (Kroupa, 1960) and in the half space (Bastecka, 1964).

In the present study, Eshelby's method for the ellipsoidal inclusion, the Hankel transformation method for the axisymmetric problems and the Mindlin's stress functions for strain nuclei of double force with moment are used for the analysis of the elastic solution of an ellipsoidal inclusion in the half space when a uniform eigenstrain with components e_{33}^T , $e_{11}^T = e_{22}^T$, e_{31}^T , e_{23}^T and $e_{12}^T = 0$ are given initially inside the inclusion. Existing solutions are shown to be special cases of the present one.

2. ELASTIC SOLUTIONS

The present problem is to express the elastic field when the eigenstrain e_{ij}^T in an ellipsoidal subdomain Ω_1 (with semi-axes a_1 , a_2 , a_3 , and center at $x_1 = x_2 = 0$ and $x_3 = c$) of the half space $x_3 > 0$ (Fig.1) is made up of components $e_{11}^T = e_{22}^T$, e_{33}^T , e_{31}^T , e_{23}^T and $e_{12}^T = 0$. In order that the plane $x_3 = 0$ be a surface free of external tractions, the stress components on this plane must satisfy the following boundary conditions

$$(\sigma_{13})_{x_3=0} = (\sigma_{23})_{x_3=0} = (\sigma_{33})_{x_3=0} = 0, \quad (1)$$

and both the equations of equilibrium

$$\sigma_{ij,j} = 0, \quad (2)$$

and the compatibility equations

$$\nabla^2 \sigma_{ij} + \frac{1}{1+\nu} \sigma_{,ij} = 0, \quad (3)$$

where the numerical suffixes, $i, j = 1, 2, 3$, following a comma denote differentiation with respect to the Cartesian coordinates x_1, x_2, x_3 , respectively; a repeated suffix is summed over values 1, 2, 3, and ν is the Poisson's ratio.

Similar to the approaches of Bastecka (1964) and Chiu (1978), the stress σ_{ij} in the half space $x_3 \geq 0$ outside the axisymmetric ellipsoidal inclusion centered at the point $(0,0,c)$ can be expressed as

$$\sigma_{ij} = \sigma_{ij}^I + \sigma_{ij}^{\Pi} + \sigma'_{ij}, \quad (4)$$

which satisfies the required boundary conditions, Eq.(1), the equilibrium condition, Eq.(2), the compatibility equations, Eq.(3), and also converges to zero for x_1 and x_2 approaching $\pm\infty$ and x_3 approaching ∞ . In Eq.(4), the term σ_{ij}^I is the stress caused by the inclusion Ω_1 centered at $(0,0,c)$ in an isotropic infinite body and can be obtained by using Eshelby's method (1961) with displacements

$$u_i^I = \frac{1}{8\pi(1-\nu)} e_{jk}^T \psi_{,ijk}^I - \frac{1}{2\pi} e_{ik}^T \phi_{,k}^I - \frac{\nu}{4\pi(1-\nu)} e_{,i}^T \phi_{,i}^I, \quad (5)$$

where ψ^I and ϕ^I , respectively, are the biharmonic and harmonic potentials of attracting matter of unit density filling the volume Ω_1 and $e^T = e_{mm}^T$ with repeat index m sum over 1, 2 and 3. σ_{ij}^{Π} is the stress caused by the image inclusion Ω_2 centered at the point $(0,0,-c)$ in an isotropic infinite body, with eigenstrain

$$(e_{ij}^T)^{\Pi} = -(e_{ij}^T)^I. \quad (6)$$

The solution for the stresses σ_{ij}^{Π} is obtained by translating the origin of coordinates in Eq.(4) from points $(0,0,c)$ to $(0,0,-c)$. The additional stress σ'_{ij} in Eq.(4) is the fictitious stress necessary to make the surface of the half-space free of stresses and it satisfies the boundary conditions

$$(\sigma'_{33})_{x_3=0} = -(\sigma_{33}^I + \sigma_{33}^{\Pi})_{x_3=0} = 0,$$

$$\begin{aligned}
(\sigma'_{13})_{x_3=0} &= -(\sigma^I_{13} + \sigma^{\Pi}_{13})_{x_3=0}, \\
(\sigma'_{23})_{x_3=0} &= -(\sigma^I_{23} + \sigma^{\Pi}_{23})_{x_3=0}.
\end{aligned} \tag{7}$$

The elastic solution of an inclusion in a half-space present in this report are solved first by considering the inclusion with pure principal eigenstrains $e^T_{11} = e^T_{22}, e^T_{33}$ and then the inclusion with pure shear eigenstrains e^T_{31}, e^T_{23} and $e^T_{12} = 0$ is solved. The final solution is the linear superposition of these solutions.

(A) Inclusion with with pure principal eigenstrains $e^T_{11} = e^T_{22}, e^T_{33}$.

The stress field of the axisymmetric penny shape inclusion with principal eigenstrains e^T_{33} in an infinite medium obtained by Eshelby method is compared with the stress field of a prismatic loop with radius a and Burger's vector b in an infinite medium as obtained by Kroupa (Yu and Sanday, 1988). A relationship is found between the potential function ϕ of the inclusion and the integral function I_0^{-1} , which involves the product of Bessel functions J_m , for the solution of the prismatic loop. That is

$$I_0^{-1} = \frac{1}{2\pi ab} e^T_{33} \phi, \tag{8}$$

where in cylindrical coordinate (r, θ, z)

$$I_m^n = \int_0^\infty t^n J_m(rt/a) J_1(t) e^{-zt/a} dt, \tag{9}$$

and J_m is the Bessel function of the m th order. In Eqs. (8) and (9), both the harmonic potential ϕ and the integral function I_0^{-1} are taken the origin as the center of the inclusion and the center of the dislocation loop, respectively.

Then the fictitious stress field is solved first for the two dimensional problem by using the Hankel transformation method and then it is transformed into the three dimensional case by use of the relationship between ϕ and I_0^{-1} . The displacements solved are

$$\begin{aligned}
u_1 &= \frac{(e_{33}^T - e_{11}^T)}{8\pi(1-\nu)} [\psi_{,133}^I - 2\nu\phi_{,1}^I + 2z\psi_{,1333}^{\Pi} + (3-4\nu)\psi_{,133}^{\Pi} \\
&\quad - 2z^2\phi_{,133}^{\Pi} - 4(2-\nu)z\phi_{,13}^{\Pi} - 2(2-3\nu)\phi_{,1}^{\Pi}] \\
&\quad - \frac{(1+\nu)e_{11}^T}{4\pi(1-\nu)} [\phi_{,1}^I + 2z\phi_{,13}^{\Pi} + (3-4\nu)\phi_{,1}^{\Pi}], \\
u_2 &= \frac{(e_{33}^T - e_{11}^T)}{8\pi(1-\nu)} [\psi_{,233}^I - 2\nu\phi_{,2}^I + 2z\psi_{,2333}^{\Pi} + (3-4\nu)\psi_{,233}^{\Pi} \\
&\quad - 2z^2\phi_{,233}^{\Pi} - 4(2-\nu)z\phi_{,23}^{\Pi} - 2(2-3\nu)\phi_{,2}^{\Pi}] \\
&\quad - \frac{(1+\nu)e_{11}^T}{4\pi(1-\nu)} [\phi_{,2}^I + 2z\phi_{,23}^{\Pi} + (3-4\nu)\phi_{,2}^{\Pi}], \\
u_3 &= \frac{(e_{33}^T - e_{11}^T)}{8\pi(1-\nu)} [\psi_{,333}^I - 2(2-\nu)\phi_{,3}^I + 2z\psi_{,3333}^{\Pi} - (3-4\nu)\psi_{,333}^{\Pi} \\
&\quad - 2z^2\phi_{,333}^{\Pi} - 4(1+\nu)z\phi_{,33}^{\Pi} + 2(4-5\nu)\phi_{,3}^{\Pi}] \\
&\quad - \frac{(1+\nu)e_{11}^T}{4\pi(1-\nu)} [\phi_{,3}^I + 2z\phi_{,33}^{\Pi} - (3-4\nu)\phi_{,3}^{\Pi}],
\end{aligned} \tag{10}$$

and the stresses are

$$\begin{aligned}
\sigma_{11} &= \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,1133}^I + 2\nu\phi_{,22}^I + 2z\psi_{,11333}^{\Pi} + (3-4\nu)\psi_{,1133}^{\Pi} - 4\nu\psi_{,3333}^{\Pi} \\
&\quad - 2z^2\phi_{,1133}^{\Pi} - 4(2-\nu)z\phi_{,113}^{\Pi} + 4\nu z\phi_{,333}^{\Pi} - 2(2-3\nu)\phi_{,11}^{\Pi} + 14\nu\phi_{,33}^{\Pi}] \\
&\quad - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,11}^I + 2z\phi_{,113}^{\Pi} + (3-4\nu)\phi_{,11}^{\Pi} - 4\nu\phi_{,33}^{\Pi}], \\
\sigma_{22} &= \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,2233}^I + 2\nu\phi_{,11}^I + 2z\psi_{,22333}^{\Pi} + (3-4\nu)\psi_{,2233}^{\Pi} - 4\nu\psi_{,3333}^{\Pi} \\
&\quad - 2z^2\phi_{,2233}^{\Pi} - 4(2-\nu)z\phi_{,223}^{\Pi} + 4\nu z\phi_{,333}^{\Pi} - 2(2-3\nu)\phi_{,22}^{\Pi} + 14\nu\phi_{,33}^{\Pi}] \\
&\quad - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,22}^I + 2z\phi_{,223}^{\Pi} + (3-4\nu)\phi_{,22}^{\Pi} - 4\nu\phi_{,33}^{\Pi}],
\end{aligned}$$

$$\begin{aligned}
\sigma_{33} = & \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,3333}^I - 4\phi_{,33}^I - \psi_{,3333}^{\Pi} + 4\phi_{,33}^{\Pi} \\
& + 2z\psi_{,3333}^{\Pi} - 2z^2\phi_{,3333}^{\Pi} - 8z\phi_{,333}^{\Pi}] \\
& - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,33}^I - \phi_{,33}^{\Pi} + 2z\phi_{,333}^{\Pi}], \\
\sigma_{12} = & \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,1233}^I - 2\nu\phi_{,12}^I + 2z\psi_{,1233}^{\Pi} + (3-4\nu)\psi_{,1233}^{\Pi} \\
& - 2z^2\phi_{,1233}^{\Pi} - 4(2-\nu)z\phi_{,123}^{\Pi} - 2(2-3\nu)\phi_{,12}^{\Pi}] \\
& - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,12}^I + 2z\phi_{,123}^{\Pi} + (3-4\nu)\phi_{,12}^{\Pi}], \\
\sigma_{23} = & \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,2333}^I - 2\phi_{,23}^I + 2z\psi_{,2333}^{\Pi} + \psi_{,2333}^{\Pi} \\
& - 2z^2\phi_{,2333}^{\Pi} - 8z\phi_{,233}^{\Pi} - 2\phi_{,23}^{\Pi}] \\
& - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,23}^I + 2z\phi_{,233}^{\Pi} + \phi_{,23}^{\Pi}], \\
\sigma_{31} = & \frac{\mu(e_{33}^T - e_{11}^T)}{4\pi(1-\nu)} [\psi_{,1333}^I - 2\phi_{,13}^I + 2z\psi_{,1333}^{\Pi} + \psi_{,1333}^{\Pi} \\
& - 2z^2\phi_{,1333}^{\Pi} - 8z\phi_{,133}^{\Pi} - 2\phi_{,13}^{\Pi}] \\
& - \frac{(1+\nu)\mu e_{11}^T}{2\pi(1-\nu)} [\phi_{,13}^I + 2z\phi_{,133}^{\Pi} + \phi_{,13}^{\Pi}],
\end{aligned} \tag{11}$$

and the dilatational stress is

$$\begin{aligned}
\sigma = & \frac{(1+\nu)\mu(e_{33}^T - e_{11}^T)}{2\pi(1-\nu)} [-\phi_{,33}^I + 7\phi_{,33}^{\Pi} - 2\psi_{,3333}^{\Pi} + 2z\phi_{,333}^{\Pi}] \\
& + \frac{2(1+\nu)^2\mu e_{11}^T}{\pi(1-\nu)} \phi_{,33}^{\Pi}.
\end{aligned} \tag{12}$$

where ψ^{Π} and ϕ^{Π} are the biharmonic and harmonic potential of attracting matter of unit density filling the volume Ω_2 , respectively.

The additional justification for the substitution of Eq. (8) is that Eshelby (1961) has shown that the remote field of a finite prismatic loop with area A and Burgers vector b is the same as the remote field of an inclusion of arbitrary shape whose volume V and the stress free strain e_{33}^T (parallel to b) satisfy

$$Ve_{33}^T = Ab, \quad (13)$$

For a circular edge dislocation loop of radius a and the x_3 -axis (or z -axis) as the axis of symmetry in an unbounded medium, the stress field is found by Kroupa (1960) by using Hankel transformations. Since for the remote field the inclusion can be of any shape, therefore, the harmonic potential ϕ for a spherical inclusion of radius a is chosen, Then Eqs. (8) and (13) give

$$I_0^{-1} = \frac{a}{2R} \quad (a \rightarrow 0), \quad (14)$$

which is the same as obtained from the mathematics formula given by Eason, Noble and Sneddon (1955). There, the substituting of Eq.(8) is not limited to penny shape inclusion, it can apply to any ellipsoidal inclusion because we can let the radius of the dislocation loop approach zero and integrate the solutions over the volume of the inclusion which gives the results in term of potential functions.

(B) Inclusion with pure shear eigenstrains e_{31}^T , e_{23}^T and $e_{12}^T = 0$.

The linear elastic solution for an inclusion with pure shear eigenstrain is obtained indirectly by the application of Kelvin's solution for double force with moment in an infinite body and Mindlin's solution for double force with moment in an semi-infinite body . A comparison between Kelvin's solution for double force with moment in an infinite body and Eshelby's solution for an inclusion with pure shear eigenstrain is conducted

first. A relation between these two solutions is established first. Then the solution for the inclusion in the half space can be obtained by substituting this relationship into the solution obtained by Mindlin for the semi-infinite solid.

For the double force with strength A in the x_1x direction with moment about x_2 axis and the double force with strength A in x_3 direction with moment about y axis in infinite solid, the Galerkin vector is (Mindlin, 1936)

$$\vec{F} = iAx_3/R + kAx_1/R, \quad (15)$$

where $R^2 = x_1^2 + x_2^2 + x_3^2$. The displacements derived from Eq. (15) are

$$\begin{aligned} u_1 &= \frac{A}{\mu} [2(1-\nu)\phi_{,3} - x_3\phi_{,11}], \\ u_2 &= \frac{A}{\mu} [x_3\phi_{,12}], \\ u_3 &= \frac{A}{\mu} [2(1-\nu)\phi_{,1} - x_3\phi_{,13}], \end{aligned} \quad (16)$$

where $\phi = 1/R$.

For penny shape inclusion (disc; $a_1 = a_2, a_3 \rightarrow 0$) with $e_{31}^T = e_{13}^T$ are the only non-zero components of e_{ij}^T . Eq. (1) gives the displacement as

$$\begin{aligned} u_1 &= -\frac{e_{13}^T}{4\pi(1-\nu)} [2(1-\nu)\phi_{,3} - x_3\phi_{,11}], \\ u_2 &= -\frac{e_{13}^T}{4\pi(1-\nu)} [x_3\phi_{,12}], \\ u_3 &= -\frac{e_{13}^T}{4\pi(1-\nu)} [2(1-\nu)\phi_{,1} - x_3\phi_{,13}], \end{aligned} \quad (17)$$

where ϕ is the harmonic potential of attracting matter of unit density filling the inclusion volume Ω centered at $(0,0,0)$ and is

$$\phi = \int_{\Omega} \phi d\tau. \quad (18)$$

Eq. (18) is valid for all shapes of ellipsoidal inclusion not just for penny shape inclusion. By comparing Eqs. (16) and (17), it is found that the solution for the inclusion with pure shear eigenstrain $e_{31}^T = e_{13}^T$ can be obtained by integrating the results of the double forces with the same strength A in both the x_1 and x_3 direction with moment about x_2 axis in an infinite solid for the strain over the volume element of the inclusion provided that

$$A = - \frac{\mu e_{13}^T}{4\pi(1-\nu)}. \quad (19)$$

For the double force at point $(0,0,c)$ with strength A in both x and z directions with moments about y axis in semi infinite solid, the Galerkin vector is (Mindlin and Cheng, 1950A)

$$\begin{aligned} \vec{F} = i & \left[\frac{x_3 - c}{R_1} - \frac{x_3 - 3c}{R_2} - \frac{2c^2(x_3 + c)}{R_2^2} - 4(1-\nu)(1-2\nu) \log(R_2 + x_3 + c) \right] \\ & \vec{k} \left[\frac{x_1}{R_1} - \frac{x_1}{R_2} + \frac{8\nu(1-\nu)x_1}{R_2} + \frac{4(1-2\nu)\{(1-\nu)x_3 - \nu c\}x_1}{R_2(R_2 + x_3 + c)} + \frac{2cx_3x_1}{R_2^3} \right. \\ & \left. + \frac{2c(x_3 + c)x_1}{R_2^3} - \frac{4(1-\nu)x_1}{R_2} \right]. \end{aligned} \quad (20)$$

Where $R_1^2 = (x_1 - c)^2 + x_2^2 + x_3^2$ and $R_2^2 = (x_1 + c)^2 + x_2^2 + x_3^2$. The stress field of these double forces can be obtained accordingly and by substituting Eq. (18), the displacement in the matrix due to the inclusion with pure shear eigenstrain $e_{31}^T = e_{13}^T$, and all others equal to 0 is found to be

$$\begin{aligned} u_1 = \frac{e_{13}^T}{4\pi(1-\nu)} & \left[\psi_{,113}^I - 2(1-\nu)\phi_{,3}^I - 2z\psi_{,1133}^{\Pi} - (3-4\nu)\psi_{,113}^{\Pi} \right. \\ & \left. + 2z^2\phi_{,113}^{\Pi} + 4(1-\nu)z\phi_{,11}^{\Pi} + 2(1-\nu)\phi_{,3}^{\Pi} \right], \\ u_2 = \frac{e_{13}^T}{4\pi(1-\nu)} & \left[\psi_{,123}^I - 2z\psi_{,1233}^{\Pi} - (3-4\nu)\psi_{,123}^{\Pi} + 2z^2\phi_{,123}^{\Pi} + 4(1-\nu)z\phi_{,12}^{\Pi} \right], \end{aligned} \quad (21)$$

$$u_3 = \frac{e_{13}^T}{4\pi(1-\nu)} \left[\psi_{,133}^I - 2(1-\nu)\phi_{,1}^I - 2z\psi_{,1333}^{\Pi} + (3-4\nu)\psi_{,133}^{\Pi} \right. \\ \left. + 2z^2\phi_{,133}^{\Pi} + 4vz\phi_{,13}^{\Pi} - 2(1-\nu)\phi_{,1}^{\Pi} \right].$$

The corresponding stress field is

$$\begin{aligned} \sigma_{11} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1113}^I - 2\phi_{,13}^I - 2z\psi_{,11133}^{\Pi} - 4v\psi_{,1223}^{\Pi} - 3\psi_{,1113}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1113}^{\Pi} + 4z\phi_{,111}^{\Pi} + 4vz\phi_{,122}^{\Pi} + 2\phi_{,13}^{\Pi} \right], \\ \sigma_{22} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1223}^I - 2v\phi_{,13}^I - 2z\psi_{,12233}^{\Pi} - 4v\psi_{,1113}^{\Pi} - 3\psi_{,1223}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1223}^{\Pi} + 4z\phi_{,122}^{\Pi} + 4vz\phi_{,111}^{\Pi} + 2v\phi_{,13}^{\Pi} \right], \\ \sigma_{33} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1333}^I - 2\phi_{,13}^I - 2z\psi_{,13333}^{\Pi} + \psi_{,1333}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1333}^{\Pi} + 4z\phi_{,133}^{\Pi} - 2\phi_{,13}^{\Pi} \right], \\ \sigma_{12} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1123}^I - (1-\nu)\phi_{,23}^I - 2z\psi_{,11233}^{\Pi} - (3-4\nu)\psi_{,1123}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1123}^{\Pi} + 4(1-\nu)z\phi_{,112}^{\Pi} + (1-\nu)\phi_{,23}^{\Pi} \right], \\ \sigma_{23} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1233}^I - (1-\nu)\phi_{,12}^I - \psi_{,1233}^{\Pi} - 2z\psi_{,12333}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1233}^{\Pi} + 4z\phi_{,123}^{\Pi} + (1-\nu)\phi_{,12}^{\Pi} \right], \\ \sigma_{31} &= \frac{\mu e_{13}^T}{2\pi(1-\nu)} \left[\psi_{,1133}^I + (1-\nu)\phi_{,22}^I - \psi_{,1133}^{\Pi} - 2z\psi_{,11333}^{\Pi} \right. \\ &\quad \left. + 2z^2\phi_{,1133}^{\Pi} + 4z\phi_{,113}^{\Pi} - (1-\nu)\phi_{,22}^{\Pi} \right], \end{aligned} \tag{22}$$

and the dilatational field is

$$\sigma = \frac{(1+\nu)\mu e_{13}^T}{\pi(1-\nu)} [-\phi_{,13}^I - 3\phi_{,13}^{\Pi} - 2z\phi_{,133}^{\Pi} + 2\psi_{,1333}^{\Pi}]. \quad (23)$$

The solution for inclusion with $e_{23}^T = e_{32}^T$, all others = 0 is found by cyclic permutation of (1,2,3). Solution Eqs.(11) and (22) can be shown to satisfy the boundary conditions, Eq. (1), equilibrium, Eq. (2) and compatibility, Eq. (3).

3. DISCUSSION

In the present analysis, the elastic field caused by an ellipsoidal inclusion was investigated for a semi-infinite solid. An infinite, isotropic elastic space was considered first with two ellipsoidal domains of the same shape with centers located at (0,0,c) and (0,0,-c). The two domains are arranged to be mirror images of each other with initial eigenstrain equal in magnitude but of opposite sign. The stress field for given initial eigenstrains in the two domains can be obtained from superposition of stresses of each single domain in an infinite solid obtained by Eshelby's method. A fictitious stress is superimposed such that the the plane $x_3 = 0$ is stress-free. This fictitious stress is obtained by the method of Hankle's transformations or by integration of the elastic solutions for nuclei of strain in the semi-infinite solid.

An important aspect that should be pointed out here is that the image field (the stress field due to the image inclusion plus the fictitious stress) which is superposed to the stress field from the single inclusion in the infinite solid in order to satisfy the free surface free of stress will change the initial eigenstrain in the inclusion. More specifically, the problem considered here was that when a initial eigenstrain of the inclusion in an infinite solid is given instead of a given initial condition in a half-space under consideration. Therefore the statements given in sections § 2.1 and § 4.1 in Eshelby's paper (1961) regarding the

sum of the original field and the image field are chosen so that the boundary conditions on the outer surface of the matrix are satisfied should be carefully interpreted. For example, consider a cavity in an half-space under external load as a special case of the ellipsoidal inhomogeneity. Then the image stresses not only have to satisfy the boundary condition at the free surface but also vanish at the inclusion-matrix interface. Consequently, when an inclusion in the half-space is considered, the image stress should satisfy not only the free surface boundary conditions, but also the boundary conditions at the inclusion-matrix interface.

It was also found out by using the relation $I_0^{-1} = a / 2R$ when $a \rightarrow 0$ (as given in Eq.(14)), we can obtain the elastic solutions of some of the nuclei of strain in the semi-infinite solid by directly application of Hankel's transformation technique and the Kelvin's solution for a force in an infinite body. Instead as shown by Mindlin and Cheng (1950A) that they are derived, by the processes of superposition, differentiation and integration, from the solution for the single force in the interior of the semi-infinite solid. The nuclei of strain solved by present approach are those axisymmetric to the x_3 axis which is normal to the free surface, such as single force in x_3 direction, double force in x_3 direction, center of dilatation, doublet with axis parallel to x_3 axis.

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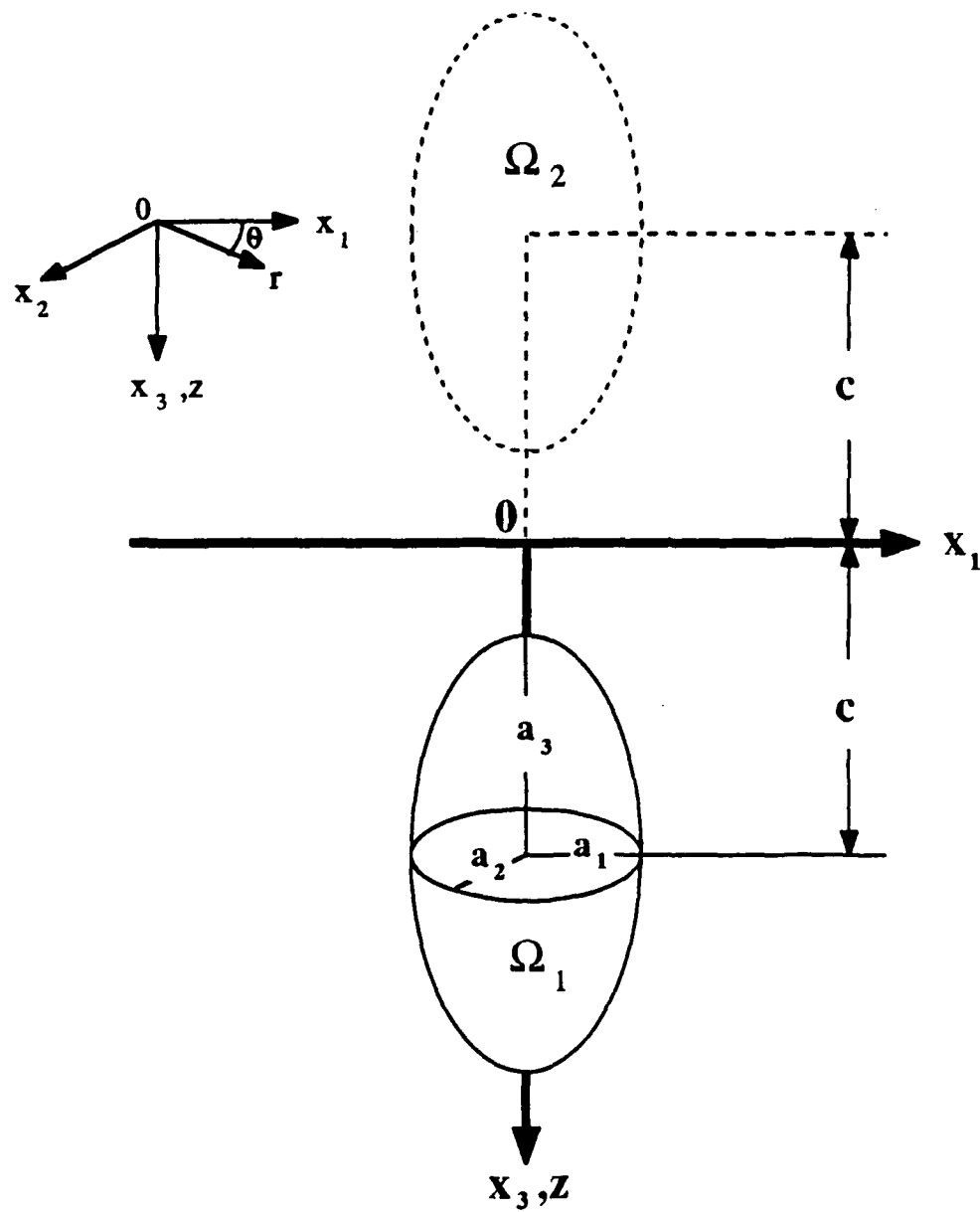


Fig. 1 - Ellipsoidal inclusion Ω_1 with principal half-axes a_1 , a_2 and a_3 in a half space and its image Ω_2 .